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STABILITY WITH RESPECT TO A SPECIFIED NUMBER OF VARIABLES*

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The stability with respect to a specified number of variables (specified quantitatively) is considered, the set of which changes depending on the initial conditions. The stability conditions of the form considered are derived using the Lyapunov-function method. The problem of the quenching of rotational motions with respect to the centre of mass of an asymptotic solid is examined; it is shown that the "twist" of a body with respect to one of the principal axes of its ellipsoid of inertia is possible using only one "fixed jet engines" for any large initial perturbations.

1. Determination of stability with respect to a specified number of variables. We will assume that the set of variables with respect to which the stability is examined is not specified in advance, and for any fairly small or even large initial perturbations it is only necessary to guarantee the stability with respect to a specified number of variables (specified quantitatively). Which of the variables will turn out to be stable is not important, and it is assumed that different variables can be stable depending on the initial conditions (which necessarily encompass the whole fairly small or even fairly large domain of the initial perturbations). More exactly, we shall introduce the following definition.

Definition 1. The unperturbed motion $\mathbf{x}=0$ of the n-dimensional system of perturbed motion

$$\mathbf{x}^{\cdot}=\mathbf{X}(t,\mathbf{x})\quad (\mathbf{X}(t,0)\equiv 0),\ \mathbf{x}=(x_1,\ldots,x_n) \tag{1.1}$$

is called stable with respect to a specified number of $m<n$ variables, if the numbers $\delta(\varepsilon,t_0)>0$ and $L>0$ are obtained for any ε, $t_0\geqslant 0$ (L does not depend on ε, t_0), such that the domain $\|\mathbf{x}_0\|<\delta$ is divided into L parts D_j, such that

$$D_1\cup D_2\cup\ldots\cup D_L=\{\mathbf{x}_0:\|\mathbf{x}_0\|<\delta\},\ D_1\cap D_2\cap\ldots\cap D_L=\{0\}$$

whilst from $\|\mathbf{x}_0\|<\delta$ it follows that

$$\|\mathbf{y}_j^+(t;\ t_0,\ \mathbf{x}_0)\|<\varepsilon\ (t\geqslant t_0)\ (j=1,\ldots,L)$$

if $\mathbf{x}_0\in D_j$. Here $\mathbf{y}_j^+=(x_{j1},\ldots,x_{jm})$ are different sets with respect to m variables from $x_1,\ldots,x_n$ (i.e. they differ in at least two elements, but not in the order of their arrangement). If, in addition, the following conditions hold:

$$\lim\|\mathbf{y}_j^+(t;\ t_0,\ \mathbf{x}_0)\|=0\ (t\to\infty)$$

the motion $\mathbf{x}=0$ of system (1.1) is asymptotically stable with respect to the specified number of variables.

Remark. 1°. We assume that the right-hand sides of system (1.1) are continuous and satisfy the conditions of uniqueness of the solution in the domain

$$\begin{gathered}t\geqslant 0,\quad \|\mathbf{y}_j^+\|\leqslant H>0,\quad 0\leqslant\|\mathbf{z}_j^+\|<+\infty\\ \mathbf{x}_j^+=(\mathbf{y}_j^+,\mathbf{z}_j^+),\quad \|\mathbf{x}\|=\|\mathbf{x}_j^+\|,\quad j=1,\ldots,L\end{gathered} \tag{1.2}$$

and any solution $\mathbf{x}(t)$ is defined for all $t\geqslant 0$, for which $\|\mathbf{y}_j^+(t)\|\leqslant H\ (j=1,\ldots,L)$.

2°. In the sense of the definition $L>2$, or when $L=1$, Definition 1 becomes a definition of y-stability in Rumyantsev's sense /1, 2/, if $D_1=D$, or becomes a definition of conditional $\mathbf{y}$-stability (in Lyapunov's sense), if $D_1\neq D$. In this connection we emphasize that when $L\geqslant 2$ the definition introduced is broader than the conditional stability with respect to part of the variables, for the whole fairly small domain of the initial perturbations is encompassed by Definition 1.

3°. The number L can equal the maximum possible number of different sets $\mathbf{y}_j^+$, although the latter is not necessary.

4°. Definition 1 permits various modifications bearing in mind the different treatments of the concept of $\mathbf{y}$-stability /2/; in particular, we can refer to uniform or exponential stability with respect to a specified number, etc..

5°. We shall also distinguish the case when the unperturbed motion $\mathbf{x}=0$ of system (1.1) is stable with respect to a specified number of variables in the whole domain Ω of variation of the parameters of the system, but the particular variables depend on the membership of the

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parameters of the system in one or other part of the domain Ω. We shall call this stability the parametric stability with respect to a specified number of variables.

2. The use of the Lyapunov function method. We shall demonstrate one of the possibilities of using the Lyapunov function method to detect stability in the sense of Definition 1.

Theorem 1. Suppose $W(\mathbf{x}), W(0) \equiv 0$ is the first integral of system (1.1).

1°. If two functions $V_j\ (j = 1, 2)$ exist, such that in the domain

$$\{t \geqslant 0, \|\mathbf{y}_j^+\| \leqslant H > 0, 0 \leqslant \|\mathbf{z}_j^+\| < +\infty\} \cap \{\mathbf{x}: (-1)^j W(\mathbf{x}) < 0\} \tag{2.1}$$

the following conditions hold:

$$V_j(t, \mathbf{x}) \geqslant a_j(\|\mathbf{y}_j^+\|), \quad V_j^{\cdot} \leqslant 0 \quad (j = 1, 2)$$

in which $a_j(r)$ are continuous functions $a_j(r)$, $a_j(0) \equiv 0$ that increase monotonically when $r \in [0, H]$, the motion $\mathbf{x} = 0$ is stable with respect to a specified number of m variables; at the same time the domain of initial perturbations $\|\mathbf{x}_0\| < \delta$ is divided into three parts: $W(\mathbf{x}_0) > 0$, $W(\mathbf{x}_0) < 0$, $W(\mathbf{x}_0) = 0$, in each of which the variables which consist of the vectors $(\mathbf{y}_1^+)$, $(\mathbf{y}_2^+)$, $(\mathbf{y}_1^+, \mathbf{y}_2^+)$. 2° respectively are stable. 2°. If, in addition, the conditions $V_j \downarrow 0$ $(j = 1, 2)$, hold, the stability with respect to the specified number of variables has an asymptotic form. (The condition $V_j \downarrow 0$ indicates that V_j approaches zero, monotonically decreasing.)

Proof. Putting $W^*(\mathbf{x}) \equiv W(\mathbf{x}) - W(\mathbf{x}_0)$, we conclude that when $\mathbf{y}_j^+ = \mathbf{x}$ the functions $V_j(t, \mathbf{x})$ when $W^*(\mathbf{x}) = 0$ satisfy the conditions of Lyapunov's theorem on conditional stability (/3, p.112/). Therefore the validity of Theorem 1 follows from Lyapunov's theorem and those of Rumyantsev /1, 2/ on stability with respect to some of the variables.

Remark. The conditions $V_j \downarrow 0$ can be verified in the same way as in the well-known theorems on stability with respect to variables /2/; for example, the condition $V_j \downarrow 0$ will hold if $V_j \leqslant b_j(\|\mathbf{y}_j^+\|)$, $V_j^{\cdot} \leqslant -c_j(\|\mathbf{y}_j^+\|)$ in the domain (2.1), where b_j, c_j are functions of the same form as a_j.

3. The problem of quenching rotational motions of a solid. Consider Euler's dynamic equations of a solid, secured at the centre of mass

$$x_1^{\cdot} = \frac{B-C}{A} x_2 x_3 + \frac{1}{A} u_1 \quad (123, ABC) \tag{3.1}$$

where $x_i\ (i = 1, 2, 3)$ are the projections of the instantaneous angular velocity of the body on the principal axes of interia, A, B, C are the principal moments of inertia, and $u_i\ (1, 2, 3)$ are the controlling moments. In particular, u_i can characterize the tractive force of the three "fixed jet engines" arranged in appropriate form (see, e.g., /4/).

Using Lyapunov's function method, it is shown in /5/ that: 1) the controls $u_i = \alpha_i x_i\ (i = 1, 2)$, $u_3 \equiv 0$ $(\alpha_i = \text{const} < 0)$ guarantee stability in Lyapunov's sense and the asymptotic (x_1, x_2)-stability of the equilibrium configuration $x_1 = x_2 = x_3 = 0$ of the closed system (3.1) for any A, B, C; 2) the control

$$u_1 = \alpha_1 x_1, \quad u_2 \equiv x_3 \equiv 0 \tag{3.2}$$

guarantees Lyapunov stability and asymptotic x_1-stability of the equilibrium configuration $x_1 = x_2 = x_3 = 0$ of system (3.1), (3.2) for any A, B, C. Below we will examine the problem of the quenching of the angular rotations of a solid with respect to two of the three variables $x_i\ (i = 1, 2, 3)$ using only one universal variable (one "fixed jet engine of low power"). We assume that the solid is asymmetric, i.e. $A \neq B \neq C$.

Theorem 2. If $B < A < C$ $(C < A < B)$, then for any small initial perturbations the equilibrium configuration $x_1 = x_2 = x_3 = 0$ of system (3.1), (3.2) is Lyapunov-stable and asymptotically stable with respect to two of the three variables $x_i\ (i = 1, 2, 3)$. At the same time the domain of initial perturbations is divided into three mutually non-intersecting parts

$$W_0 > 0 \ (W_0 < 0), \quad W_0 < 0 \ (W_0 > 0), \quad W_0 = 0 \tag{3.3}$$
$$\left(W_0 = \frac{C-A}{B} x_{30}^2 - \frac{A-B}{C} x_{20}^2\right)$$

in which the variables (x_1, x_2), (x_1, x_3), (x_1, x_2, x_3) are asymptotically stable.

Proof. We shall use the rules of /6/, which enable us to reduce the examination of stability with respect to some of the variables to an examination of stability in Lyapunov's sense for some subsidiary set of equations of μ-form. For this we shall introduce the new variable $\mu_1 = (B - C) x_2 x_3 / A$, as a result of which system (3.1), (3.2) is transformed in the following way:

$$x_1^{\cdot} = \alpha_1{}^* x_1 + \mu_1,\ \ \mu_1^{\cdot} = x_1 X(x_2,\ x_3) \tag{3.4}$$
$$x_2^{\cdot} = \frac{C-A}{B} x_1 x_3,\ \ x_3^{\cdot} = \frac{A-B}{C} x_1 x_2,\ \ \alpha_1{}^* = \frac{\alpha_1}{A}$$
$$X(x_2,\ x_3) = (B-C)\,[C\,(C-A)\,x_3^2 + B\,(A-B)\ x_2^2]$$
$$(ABC)^{-1} = \Gamma\ (t)$$

Consider the behaviour of the function $\Gamma(t)$ along the trajectories of system (3.1), (3.2). Note that system (3.1), (3.2) allows of the first integral

$$W = \frac{C-A}{B} x_3^2 - \frac{A-B}{C} x_2^2 = W_0 = \text{const} \tag{3.5}$$

We will first assume that $W_0 \neq 0$, and shall show that when the condition $C < A < B$ or $B < A < C$ holds along the trajectories of system (3.1), (3.2) we have the following inequality:

$$\Gamma\ (t) \leqslant -\gamma_0 = \text{const} < 0 \tag{3.6}$$

Assuming, on the contrary, that $\lim \Gamma(t) = 0\ (t \to \infty)$ or a finite instant of time $t = t_*$ exists, for which $\Gamma(t_*) = 0$, and bearing in mind that $X(x_2,\ x_3)$ is a negative-definite function for all $x_2,\ x_3$, we conclude that $\lim x_i^2(t) = 0\ (t \to \infty)\ (i = 2, 3)$ or $x_2(t_*) = x_3(t_*) = 0$ and, consequently, in Eq.(3.5) $\lim W(t) = 0\ (t \to \infty)$ or $W = 0$ when $t = t_*$, which is impossible. This means that inequality (3.6) holds for all $t \geqslant t_0$.

We can now write the first two Eqs.(3.4) in the following way (a system of μ-form /6/): $x_1^{\cdot} = \alpha_1{}^* x_1 + \mu_1,\ \mu_1^{\cdot} = \Gamma(t)\, x_1$ or in the form of the equation

$$x_1^{\cdot\cdot} + p x_1^{\cdot} + q\,(t) x_1 = 0\ \ (p = -\alpha_1{}^*,\ \ q\,(t) = -\Gamma\,(t) \geqslant \gamma_0) \tag{3.7}$$

It is well-known (/7, p.238/ or /8, p.255/), that when the following condition holds:

$$p > \sqrt{\Gamma_0} - \sqrt{\gamma_0}\ \ (p \geqslant {}^1/_2\, \varepsilon q - {}^1/_2\, q^{\cdot}/q,\ \ \varepsilon > 0) \tag{3.8}$$

where $\Gamma_0 = \text{const} > 0$ is the upper limit of the function $q\,(t)$ (i.e. $\gamma_0 \leqslant q\,(t) \leqslant \Gamma_0$), the solution $x_1 = x_1^{\cdot} = 0$ of Eq.(3.7) is asymptotically stable in Lyapunov's sense. We shall verify the possibility of satisfying condition (3.8) in the above case. Note that in view of the non-asymptotic stability, in Lyapunov's sense, of the equilibrium configuration $x_1 = x_2 = x_3 = 0$ of system (3.1), (3.2) the following estimate will hold in a fiarly small neighbourhood of the origin of coordinates:

$$\sqrt{\Gamma_0} - \sqrt{\gamma_0} < \varepsilon_1$$
$$\left(\left|-\frac{q^{\cdot}}{2q}\right| = \left|-\frac{\Gamma^{\cdot}}{2\Gamma}\right| = \left|\left(\frac{\partial X}{\partial x_2}\frac{C-A}{B} x_1 x_3 + \frac{\partial X}{\partial x_3}\frac{A-B}{C} x_1 x_2\right)(2X(x_2, x_3))^{-1}\right| < \varepsilon_1\right)$$

where ε_1 is a fairly small number which is specified in advance. This means, for small initial perturbations (and for fairly small $\varepsilon_1 > 0$) inequality (3.8) holds for any fixed number $p = -\alpha_1{}^* > 0$.

The solution $x_1(t),\ x_1^{\cdot}(t)$ of Eq.(3.7) are simultaneously also solutions $x_1(t),\ \mu_1(t)$ of the initial system (3.1), (3.2); this indicates that the equilibrium configuration $x_1 = x_2 = x_3 = 0$ of system (3.1), (3.2) has the following properties:

1) $\delta > 0$ will be obtained for any $\varepsilon_1,\ t_0 \geqslant 0$, such that from $|x_{10}| < \delta,\ |x_{20} x_{30}| < \delta$ for all $t \geqslant t_0$ follows

$$|x_1(t;\ t_0,\ \mathbf{x}_0)| < \varepsilon,\ \ |x_2(t;\ t_0,\ \mathbf{x}_0)\, x_3(t;\ t_0,\ \mathbf{x}_0)| < \varepsilon \tag{3.9}$$

2) $\Delta(t_0) > 0$ will be obtained for each $t_0 \geqslant 0$, such that when $|x_{10}| < \Delta,\ |x_{20} x_{30}| < \Delta$

$$\lim |x_1(t)| = 0,\ \ \lim |x_2(t)\, x_3(t)| = 0\ \ \ \ (t \to \infty) \tag{3.10}$$

We shall show that the following relations will follows from conditions (3.9), (3.10):

$$|x_\alpha(t)| < \varepsilon,\ \lim |x_\alpha(t)| = 0\ \ (t \to \infty)$$

where $\alpha = 2$ or $\alpha = 3$ depending on the initial conditions.

To determine the geometric limits of the convergence of the solutions of system (3.1), (3.2) we shall put

$$x_2^2(t)\, x_3^2(t) = f(t) \tag{3.11}$$

and, omitting the intermediate calculations, we obtain the following solutions from Eqs.(3.5), (3.11):

$$x_3^2(t) = \frac{BW_0}{2(C-A)} + \left[\frac{B^2 W_0^2}{4(C-A)^2} + \frac{(A-B)\,Bf(t)}{C\,(C-A)}\right]^{1/2} \tag{3.12}$$
$$x_2^2(t) = -\frac{CW_0}{2(A-B)} + \left[\frac{C^2 W_0^2}{4(A-B)^2} + \frac{(C-A)\,Cf(t)}{B(A-B)}\right]^{1/2}$$

(we discard solutions with a minus sign in front of the radical).

We can now verify that when $B < A < C$ $(C < A < B)$ for $W_0 < 0$ the relation $x_3^2(t) \to 0$ $(x_2^2(t) \to 0)$ holds, and when $W_0 > 0$ the relation $x_2^2(t) \to 0$ $(x_3^2(t) \to 0)$ holds. This means that when $W_0 \neq 0$ the statement of Theorem 2 is proved.

We shall separately examine the case when $W_0 = 0$, when a finite instant of time $t = t_*$ is possible, such that $\Gamma(t_*) = 0$; the asymptotic relation $\lim \Gamma(t) = 0$ $(t \to \infty)$ indicates the Lyapunov asymptotic stability of the equilibrium configuration $x_1 = x_2 = x_3 = 0$ of system (3.1), (3.2). Since the condition $\Gamma(t_*) = 0$ is equivalent to the conditions $x_2(t_*) = x_3(t_*) = 0$ and, in addition, system (3.1), (3.2) has the solution $x_1 = x_1(t; t_0, x_0)$, $x_2 \equiv x_3 \equiv 0$, conditions $x_1(t_*) \neq 0$, $x_2(t_*) \equiv x_3(t_*) \equiv 0$ will determine the solution $x_1 = x_1(t; t_*, x_1(t_*))$, $x_2(t) \equiv x_3(t) \equiv 0$ for all $t \geqslant t_*$. By virtue of the feasibility for system (3.1), (3.2) of the conditions of /9/, which guarantee the uniqueness of the solutions, this solution will be unique for the above initial conditions. The equilibrium configuration $x_1 = x_2 = x_3 = 0$ of system (3.1), (3.2) is (non-asymptotically) Lyapunov-stable, and the function $x_1(t; t_*, x_1(t_*))$ satisfies the differential equation $x_1^{\cdot} = \alpha_1^* x_1$ $(\alpha_1^* = \text{const} < 0)$ when $t \geqslant t_*$; therefore the asymptotic stability, in Lyapunov's sense, of the equilibrium configuration $x_1 = x_2 = x_3 = 0$ of system (3.1), (3.2), follows from the conditions $W_0 = 0, \Gamma(t_*) = 0$. When $W_0 = 0$, $\Gamma(t) \leqslant -\Gamma_0 = \text{const} < 0$ a similar property of the equilibrium configuration of system (3.1), (3.2) follows from Eq.(3.12). The theorem is proved.

Remark. The condition $V \downarrow 0$ (/8, p.255/) holds for Lyapunov's function

$$V = x_1^2 + \frac{x_1^{\cdot 2}}{q(t)} = x_1^2 + \frac{(B-C)^2}{A^2 q(t)} x_2^2 x_3^2$$

along the trajectories of Eq.(3.7), and consequently also along the trajectories of system (3.1), (3.2). Bearing in mind that the integral (3.5) also occurs for Eqs.(3.1), (3.2), instead of the function V we can consider the two functions:

$$V_1 = x_1^2 + \frac{(B-C)^2}{A^2 q(t)} \frac{B}{C-A} W_0 x_2^2 + \frac{(B-C)^2 (A-B) B}{A^2 (C-A) C q(t)} x_2^4$$

$$V_2 = x_1^2 + \frac{(B-C)^2}{A^2 q(t)} \frac{C}{A-B} (-W_0) x_3^2 + \frac{(B-C)(C-A) C}{A^2 (A-B) B q(t)} x_3^4$$

We can verify that in the domain

$$\{t \geqslant 0, \|\mathbf{y}_j^+\| \leqslant H, \|\mathbf{z}_j^+\| < +\infty\} \cap \{W(-1)^j < 0\}$$
$$\mathbf{y}_1^+ = (x_1, x_2), \quad \mathbf{y}_2^+ = (x_2, x_3), \quad j = 1, 2$$

the functions V_j $(j = 1, 2)$, W satisfy all the conditions of Theorem 1.

We shall show that, choosing the quantity $|\alpha_1|$ in (3.2) to be fairly large, we can achieve the satisfaction of the conditions of Theorem 2 for any previously specified domain of the initial conditions.

Theorem 3. If $B < A < C$ $(C < A < B)$, the equilibrium configuration $x_1 = x_2 = x_3 = 0$ of system (3.1), (3.2) is also asymptotically Lyapunov stable with respect to two of the three variables x_i $(i = 1, 2, 3)$ in the domain

$$\sqrt{R_0} \leqslant -\alpha_1^* / \sqrt{l}, \quad R_0 = A x_{10}^2 + B x_{20}^2 + C x_{30}^2$$
$$l = \max\left\{\frac{(C-B)(C-A)}{ABC}, \frac{(C-B)(A-B)}{ABC}\right\} \tag{3.13}$$

As the same time the domain (3.13) is divided into three mutually non-intersecting parts (3.3), in which the variables (x_1, x_2), (x_1, x_3), (x_1, x_2, x_3) respectively are asymptotically stable.

Proof. It is well-known (/7, p.238/), that if for all t, x_i $(i = 1, 2, 3)$ the following inequality holds:

$$-\alpha_1^* > \sqrt{\Gamma_0} - \sqrt{\gamma_0} \tag{3.14}$$

then the solution $x_1 = x_1^{\cdot} = 0$ of Eq.(3.7) is asymptotically stable with respect to both variables for any initial perturbations. We shall express inequality (3.14) in terms of the initial conditions x_{i0} $(i = 1, 2, 3)$. For this we note that the function $R = A x_1^2 + B x_2^2 + C x_3^2$ satisfies the equation $R^{\cdot} = 2\alpha_1^* A x_1^2$ by virtue of system (3.1), (3.2), and consequently is non-increasing for all $t \geqslant t_0$. This means that for all $t \geqslant t_0$

$$R \leqslant R_0 \tag{3.15}$$

In addition, the following estimate holds for the function $\Gamma(t)$

$$-\Gamma(t) = (C-B)\ [C(C-A) x_3^2(t) + B(A-B) x_2^2(t)]$$
$$(ABC)^{-1} \leqslant l (C x_3^2 + B x_2^2) \leqslant lR \tag{3.16}$$

From (3.15), (3.16) we conclude that $-\Gamma(t) \leqslant lR_0$, and, consequently, $\Gamma_0 = lR_0$.

Since the inequality (3.14) will hold if $-\alpha_1^* > \sqrt{\Gamma_0}$, then when the initial perturbations $x_{i0}\ (i = 1, 2, 3)$ in system (3.1), (3.2) satisfy the condition $-\alpha_1 > \sqrt{lR_0}$, the solution $x_1 = x_1^{\cdot} = 0$ of Eq.(3.7) is asymptotically stable with respect to $x_1, x_1^{\cdot}$, and consequently the equilibrium configuration $x_1 = x_2 = x_3 = 0$ of system (3.1), (3.2) is asymptotically stable with respect to $x_1, (B - C)\, x_2x_3/A$. Further proof is carried out using the same scheme as that of Theorem 2.

Theorem 4. Suppose the solutions of system (3.1), (3.2), which begin in the fairly small neighbourhood of the point $x_1 = x_2 = x_3 = 0$, are bounded for fairly small perturbing moments $M_i\ (i = 1, 2, 3)$ with respect to the principal axes of inertia of the body. If $B < A < C\ (C < A < B)$, then for any small initial perturbations the equilibrium configuration $x_1 = x_2 = x_3 = 0$ of the closed system (3.1), (3.2) is stable in Lyapunov's sense and stable with respect to two of the three variables $x_i\ (i = 1, 2, 3)$ for constantly active small perturbing moments $M_i\ (i = 1, 2, 3)$. At the same time the domain of the initial perturbations is divided into three mutually non-intersecting parts (3.3), in which the variables $(x_1, x_2), (x_1, x_3), (x_1, x_2, x_3)$ respectively are stable.

Proof. In the case $M_i \neq 0\ (i = 1, 2, 3)$, after introducing the new variable $\mu_1 = (B - C)\, x_2x_3/A$ Eqs.(3.1), (3.2) are transformed in the following way:

$$x_1^{\cdot} = \alpha_1^* x_1 + \mu_1 + M_1$$

$$\mu_1^{\cdot} = \Gamma(t)\, x_1 + \Lambda, \quad \Lambda = \frac{B - C}{A}(x_2M_3 + x_3M_2)$$

On the assumption made about the boundedness of the solutions for all $t \geqslant t_0$, we have $|\Lambda| < \delta$, where δ is a fairly small number, if M_2, M_3 are fairly small. Since the asymptotic stability of the solution $x_1 = x_1^{\cdot} = 0$ of Eq.(3.7) is established in /7/ using a Lyapunov function that is not time-dependent and whose derivative is negative-definite, the solution $x_1 = x_1^{\cdot} = 0$ of Eq.(3.7) is stable with constantly acting perturbations on the basis of Maklin's theorem /10/. This means that for fairly small perturbing moments $M_i\ (i = 1, 2, 3)$ the equilibrium configuration $x_1 = x_2 = x_3 = 0$ of system (3.1), (3.2) is stable with respect to $x_1, (B - C)\, x_2x_3/A$. Using the procedure for proving Theorem 2, we conclude that the inequality $|x_2| < \varepsilon$ or $|x_3| < \varepsilon$ follows from the condition $(B - C)^2x_2^2x_3^2\, |\, A^2 < \varepsilon$, depending on the sign of W_0.

4. Practical application of the results obtained.

1°. The conditions of Theorem 3 guarantee a "twist" of the body with respect to the larger or smaller (depending on the value of the initial perturbations) axis of its ellipsoid of inertia for any initial perturbations. At the same time the angular velocity ω^2 with which the body rotates after the twist is determined by the equation

$$\omega^2 = W_0/E \tag{4.1}$$

where $E = (C - A)/B$ or $E = (B - A)/C$ depending on the axis around which the twist occurs. As follows from (4.1), the angular velocity ω^2 does not depend on x_{10} – the projection of the initial angular velocity of the body on the axis, with respect to which it gives the "fixed jet engine" moment, and decreases in comparison with the initial value $x_{\alpha 0}\ (\alpha = 2$ or $\alpha = 3)$ to a value $E_1 = C(C - A)\, x_{30}^2/[(B - A)\, B]\ (\alpha = 2)$ or $E_2 = B(A - B)\, x_{20}^2/[(C - A)\, C]\ (\alpha = 3)$. Therefore whether or not the angular velocity of the body after a twist attains a value dictated by the initial technical requirement depends on the quantities x_{20}, E_1 and x_{30}, E_2. The twist of the body is of interest in a number of problems of space-flight apparatus control /11/, and also at the intermediate stage of the active control of the rotational motion of a solid, for, after the twist occurs, we can construct control laws /12, 13/ that turn the body at a specified angle and stop its rotation completely.

2°. It is known /14, 15/ that the preliminary twist o of the satellite around the major axis of the ellipsoid of inertia is used for passive stabilization of satellites. Since the angular motion of a satellite is described by Eqs.(3.1) (/8, p.256/) when the natural angular velocity of a satellite significantly exceeds its angular velocity of revolution round the orbit, then in the case of a fairly large value of (4.1) we can effect the satellite twist using all or only one fixed jet engine.

3°. The situation called in Par.1 parametric stability with respect to a specified number of variables arises, for example, when examining the stability of Lotki-Volterra systems (/3, p.207/). The concept introduced can be used to formulate accurately the Lotki-Volterra ecological principle of extinction.

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ON EVOLUTIONARY MOTIONS OF A PARTICLE IN A GRAVITATIONAL FIELD*

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Non-conservative, time-periodic perturbations of the Kepler problem are studied. A phase-averaged system is given, which determines the evolution in the system when there are no resonance modes. The qualitative behaviour of the solutions in the resonance zones is studied. Depending on the structure of the behaviour of the solutions, the resonances are divided into traversable, partially traversable and non-traversable. The boundedness of the set of partially traversable resonances is established, and this, in many cases, makes it possible to determine evolution in a system with resonance modes. An example is used to illustrate the method. It is shown that a constant component in the periodic function of the perturbation causes the evolutionary process to become non-unidirectional.

1. Formulation of the problem. Consider the motion of a "particle" in a gravity field in a medium whose resistances R depend periodically on time. If r, φ are polar coordinates in the orbital plane, then the normal and tangential component of the resistance force is equal to $-mRr^{\cdot}/v$, $-mR\varphi^{\cdot}r/v$ respectively. Here m is the mass of the particle, R is the resistance per unit mass of the particle and v is the orbital velocity /1/. We write $R = \varepsilon g(r, v, \Omega t)$ where ε is a small positive parameter, the function g is at least continuous in t and periodic in Ωt with period 2π, Ω is the perturbation frequency. We also assume that g is analytic in r and $v(r)$ in the region $r \geqslant r_- > 0$. The equations of motion of the particle can be written in the form

$$r^{\cdot\cdot} - \alpha^2/r^3 + M/r^2 = -\varepsilon g r^{\cdot}/v,\ \alpha^{\cdot} = -\varepsilon\alpha g/v \tag{1.1}$$

where $\alpha = r^2\varphi^{\cdot}$ is the kinetic moment of the particle, $M = G(m_0 + m)$, m_0 is the mass of the central body and G is the gravitational constant.

A characteristic feature of system (1.1) is the resonances, i.e. the integer-type relations connecting the perturbation frequency Ω with the characteristic frequency (with mean angular motion ω):

$$p\omega = q\Omega \tag{1.2}$$

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